A CONTINUOUS MODEL OF TRANSPORTATION REVISITED

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Abstract. We review two models of optimal transport, where congestion effects during the transport can be possibly taken into account. The first model is Beckmann's one, where the transport activities are modeled by vector fields with given divergence. The second one is the model by Carlier et al. (SIAM J Control Optim 47: 1330–1350, 2008), which in turn is the continuous reformulation of Wardrop's model on graphs. We discuss the extensions of these models to their natural functional analytic setting and show that they are indeed equivalent, using an *ad hoc* generalization of Smirnov decomposition theorem for flat $1-$currents, which may fail to be normal.

Contents

1. Introduction

1.1. **Theoretical background.** The present work is motivated by the study of transport problems for distributions, started in [6] and [12], which we want to try and connect to related works in the theory of currents present in [1, 27, 36]. A motivation for such an

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extension is the study of distributions of the form

$$\sum_{i=1}^{\infty}(\delta_{P_i}-\delta_{Q_i}), \qquad \text{with} \qquad \sum_{i=1}^{\infty}|P_i-Q_i|<\infty,$$

as in [33]. Such distributions arise as topological singularities in several geometric variational problems as described for example in [11, 23, 32, 34, 35].

To start with, we formally define three variational problems which can be settled (for simplicity) on the closure of an open convex subset $\Omega\subset\mathbb{R}^N$, having smooth boundary. For the moment, we will be a little bit imprecise about the datum f, but we will properly settle our hypotheses later. The first problem is the minimization of the total variation of a Radon vector measure, under a divergence constraint:

$$(\mathcal{B}) \qquad \min_V\left\{\int_\Omega d|V|\,:\,-\operatorname{div} V=f,\ V\cdot\nu_\Omega=0\right\}.$$

The above problem can be connected by duality with the following one, called the Kantorovich problem:

$$(\mathcal{K}) \qquad \max_\phi\left\{\langle f,\phi\rangle:\|\nabla\phi\|_{L^\infty(\Omega)}\le 1\right\},$$

where now the variable ϕ is a Lipschitz function and $\langle\cdot,\cdot\rangle$ represents a suitable duality pairing. Finally, the third problem is the minimization of the total length

$$(\mathcal{M}) \qquad \min_Q\left\{\int_{\mathcal{P}}\ell(\gamma)\,dQ(\gamma):\,(e_0-e_1)_\#Q=f\right\},$$

where $\mathcal{P}$ is the space of Lipschitz continuous paths $\gamma:[0,1]\to\Omega$, the *length functional* ℓ is defined by

$$\ell(\gamma)=\int_0^1|\gamma'(t)|\,dt,$$

e_0,e_1 are the evaluation functions giving the starting and ending points of a path, and the variable Q is a measure concentrated on $\mathcal{P}$.

The classical setting for the above problems is when f is of the form $f=f^+-f^-$, where f^+ and f^- are positive measures on Ω having the same mass (for example, conventionally one can consider them to be probability measures). We point out that in this case, a more familiar formulation of $(\mathcal{M})$ is certainly the so-called *Monge-Kantorovich problem*

$$(\mathcal{M}') \qquad \min_\eta\left\{\int_{\Omega\times\Omega}|x-y|\,d\eta(x,y)\,:\,(\pi_x)_\#\eta=f^+ \quad \text{and} \quad (\pi_y)_\#\eta=f^-\right\},$$

where $\pi_x,\pi_y:\Omega\times\Omega\to\Omega$ stand for the projections on the first and second variable, respectively. It is useful to recall that the link between $(\mathcal{M})$ and $(\mathcal{M}')$ is given by the fact that if η_0 is optimal for Monge-Kantorovich problem then the measure which concentrates on *transport rays*, i.e.

$$Q_0=\int\delta_{\overline{xy}}\,d\eta_0(x,y), \qquad \text{where } \overline{xy} \text{ stands for the segment connecting } x \text{ and } y,$$

is optimal in $(\mathcal{M})$ and

$$\int_{\mathcal{P}} \ell(\gamma)\, dQ_0(\gamma) = \int_{\Omega\times\Omega} |x-y|\, d\eta_0(x,y).$$

When f has the above mentioned form $f^+ - f^-$, the equivalence of the three problems above is well understood: the equivalence of $(\mathcal{M}')$=$(\mathcal{M})$ and $(\mathcal{K})$ is the classical Kantorovich duality (see [24]), while that between $(\mathcal{B})$ and $(\mathcal{K})$ seems to have been first identified in [37].

Recently the equivalence of the above three problems has been shown in [6] for f belonging to a wider class, i.e. when f is in the completion of the space of zero-average measures with respect to the norm dual to the C^1 (or flat) norm. This wider space was studied in [22] and characterized recently in [6, 7]. A different point of view is also available in [25], where the space of such f is called $W^{-1,1}$.

1.2. **Goals of the paper.** Our starting observation is that problem $(\mathcal{B})$ pertains to a wide class of optimal transport problem, introduced by Martin J. Beckmann in [2], which are of the form

$$(\mathcal{B}_\mathcal{H}) \quad \min_V \left\{ \int_\Omega \mathcal{H}(V)\, dx \,:\, \operatorname{div} V = f,\, V\cdot \nu_\Omega = 0 \right\}, \qquad \text{where} \quad c_1\,|z|^p \le \mathcal{H}(z) \le c_2\,|z|^p,$$

for a suitable density-cost convex function $\mathcal{H} : \mathbb{R}^N \to \mathbb{R}^+$ and $p \ge 1$. For a problem of this type, the question of finding equivalent formulations of the form $(\mathcal{K})$ and $(\mathcal{M})$ has already been addressed in [10] (see also [8]), under some restrictive assumptions on f, like for example

$$f = f^+ - f^- \qquad \text{with} \quad f^+, f^- \in L^p(\Omega) \quad \text{and} \quad \int_\Omega f^+ = \int_\Omega f^- = 0.$$

The goal of this paper is to complement and refine this analysis, first of all by studying problem $(\mathcal{B}_\mathcal{H})$ in its natural functional analytic setting, i.e. when f belongs to some dual Sobolev space $W^{-1,p}$ (whose elements are not measures, in general). Also, by expanding the analysis in [8, 10], we will see that alternative formulations of the type $(\mathcal{K})$ and $(\mathcal{M})$ are still possible for $(\mathcal{B}_\mathcal{H})$ in this extended setting. These formulations are still well-posed on the dual space $W^{-1,p}$ and equivalence can be proved in this larger space. The problem corresponding to $(\mathcal{K})$ will now have the form (see Section 3 for more details)

$$(\mathcal{K}_\mathcal{H}) \qquad \max_\phi \left[\langle f, \phi\rangle - \int_\Omega \mathcal{H}^*(\nabla\phi)\, dx \right],$$

and the equivalence with $(\mathcal{B}_\mathcal{H})$ will just follow by standard convex duality arguments (that we repeat in this paper, for reader's convenience). On the contrary, in the proof of the equivalence between $(\mathcal{B}_\mathcal{H})$ and its Lagrangian formulation

$$(\mathcal{M}_\mathcal{H}) \qquad \min_Q \left\{ \int_\Omega \mathcal{H}(i_Q)\, dx \,:\, (e_0 - e_1)_\# Q = f \right\},$$

some care is needed, due to the fact that f now is not a measure. Here the measure i_Q will be some sort of *transport density*[1] generated by Q, which takes into account the amount of work generated in each region by our distribution of curves Q (see Section 4 for the precise definition). In particular, the proof of this equivalence will point out another not emphasized connection to Geometric Measure Theory.

The main result of this paper can be formulated as follows (see Theorems 3.1 and 4.5 for more precise statements):

Main Theorem. *Let $1 < p < \infty$. Suppose $\Omega \subset \mathbb{R}^N$ is the closure of a smooth bounded open set, let $f \in W^{-1,p}(\Omega)$ and let $\mathcal{H}$ be a strictly convex function having $p-$growth, with $p > 1$. Then the minima in ($\mathcal{B}_\mathcal{H}$), ($\mathcal{M}_\mathcal{H}$) and the maximum in ($\mathcal{K}_\mathcal{H}$) are achieved and coincide. Moreover we have the following relationship among the optimizers of the three problems:*

(i) *the unique minimizer of ($\mathcal{B}_\mathcal{H}$) corresponds to a minimizer of ($\mathcal{M}_\mathcal{H}$), in the sense of Proposition 4.3;*

(ii) *each minimizer of ($\mathcal{M}_\mathcal{H}$) corresponds to the unique minimizer of ($\mathcal{B}_\mathcal{H}$), in the sense of Proposition 4.3;*

(iii) *the unique minimizer V of ($\mathcal{B}_\mathcal{H}$) and the unique maximizer v of ($\mathcal{B}_\mathcal{H}$) are linked by the relation $V(x) \in \partial\mathcal{H}^*(x, \nabla v(x))$ a.e. in Ω, as specified in Theorem 3.1.*

The connection of the first two points in the above theorem to Geometric Measure Theory lies in the basic theory of *flat* 1-*currents*, of which we recall the first steps in the (long) appendix at the end of the paper. Indeed, in order to show equivalence of ($\mathcal{B}_\mathcal{H}$) and ($\mathcal{M}_\mathcal{H}$) in full generality, i.e. in the space of distributions $W^{-1,p}$, the cornerstone will be *Smirnov decomposition theorem* for 1−currents. We also notice that to this aim we need to slightly extend Smirnov's result to cover the case of flat currents with finite mass, i.e. we drop the assumption on the finiteness of the mass of their boundaries (see Theorem A.20). Translating this in the language of vector fields V, this extension is needed since the divergence of V now will be just an element of $W^{-1,p}$, rather than a measure.

For the sake of completeness and in order to neatly motivate the studies performed in this paper, it is worth recalling that the proof of this equivalence in [10] was based on the *Dacorogna-Moser construction* to produce transport maps (see [13]), which has been revealed a powerful tool for optimal transport problems[2]. In a nutshell, this method consists in associating to the "static" vector field V admissible in ($\mathcal{B}_\mathcal{H}$), the following dynamical system

$$\partial\mu_t = \operatorname{div}\left(\frac{V}{(1-t)\,f^+ + t\,f^-}\,\mu_t\right), \qquad \mu_0 = f^+,$$

[1]When $\mathcal{H}(t) = |t|$, problem ($\mathcal{M}_\mathcal{H}$) is again the Monge-Kantorovich one and i_Q for an optimal Q is nothing but the usual concept of transport density, see [5, 15, 20].

[2]It is worth remarking that the first proof of the existence of an optimal transport map for problem ($\mathcal{M}'$), more than 200 years after Monge stated it, was based on a clever use of this construction (see [18]).

i.e. a continuity equation with driving velocity field $\widetilde{V}_t$ given by V rescaled by the linear interpolation between f^+ and f^-. Supposing that one can give a sense (either deterministic or probabilistic) to the flow of $\widetilde{V}_t$, then the construction of the measure Q_V concentrated on the flow lines of $\widetilde{V}_t$ paves the way to the equivalence between the Lagrangian model $(\mathcal{M}_{\mathcal{H}})$ and $(\mathcal{B}_{\mathcal{H}})$ (see [8, 10] for more details). However, this strategy seems to need some restriction on the datum f, in particular f should be a Radon measure on Ω.

1.3. **Plan of the paper.** In Section 2 we describe the function spaces $W^{-1,p}(\Omega)$ and prove the existence of a minimizer for $(\mathcal{B}_{\mathcal{H}})$. In Section 3 we prove the equivalence of $(\mathcal{B}_{\mathcal{H}})$ with $(\mathcal{K}_{\mathcal{H}})$, by appealing to classical convex analysis formulas. The aim Section 4 is to introduce the Lagrangian counterpart of Beckmann model and to show how the two models turn out to be equivalent. A self-contained Appendix complements the paper. There we introduce relevant concepts from Geometric Measure Theory and prove the extension of Smirnov's decomposition theorem to general flat 1−currents with finite mass (Theorem A.20), which is the main ingredient for the proofs of Section 4.

2. Well-posedness of Beckmann problem

Let $\Omega \subset \mathbb{R}^N$ be the closure of an open bounded connected set, having smooth boundary. In particular, in what follows Ω *will always be compact*. Given $1 < q < \infty$, we indicate with $W^{1,q}(\Omega)$ the usual Sobolev space of $L^q(\Omega)$ functions, whose distributional gradient is in $L^q(\Omega)$ as well. We then define the quotient space

$$\dot{W}^{1,q}(\Omega) = \frac{W^{1,q}(\Omega)}{\sim},$$

where $\sim$ is the equivalence relation defined by

$$u \sim v \qquad \Longleftrightarrow \qquad \text{there exists } c \in \mathbb{R} \text{ such that} \qquad u(x) - v(x) = c \in \mathbb{R} \qquad \text{for a.e. } x \in \Omega.$$

When needed, the elements of $\dot{W}^{1,q}(\Omega)$ will be identified with functions in $W^{1,q}(\Omega)$ having zero mean. We endow the space $\dot{W}^{1,q}(\Omega)$ with the norm

$$\|u\|_{\dot{W}^{1,q}(\Omega)} := \left(\int_\Omega |\nabla u(x)|^q\,dx\right)^{\frac{1}{q}}, \qquad \dot{u} \in W^{1,q}(\Omega),$$

then we denote by $\dot{W}^{-1,p}(\Omega)$ its dual space, equipped with the dual norm, defined as usual by

$$\|T\|_{\dot{W}^{-1,p}(\Omega)} := \sup\left\{\langle T, \varphi\rangle \,:\, \varphi \in \dot{W}^{1,q}(\Omega),\ \|\varphi\|_{\dot{W}^{1,q}} = 1\right\},$$

where $p = q/(q-1)$. We start recalling the following basic fact.

Lemma 2.1. *Let $T \in \dot{W}^{-1,p}(\Omega)$, then*

$$\|T\|_{\dot{W}^{-1,p}(\Omega)} = p^{\frac{1}{p}}\left[\max_{\varphi\in\dot{W}^{1,q}(\Omega)} |\langle T, \varphi\rangle| - \frac{1}{q}\int_\Omega |\nabla\varphi|^q\,dx\right]^{\frac{1}{p}}.$$

Proof. For every $\varphi \in \dot W^{1,q}(\Omega)$, we clearly have

$$|\langle T, \varphi\rangle| - \frac{1}{q}\int_\Omega |\nabla \varphi|^q\,dx \le \sup_{\lambda\ge 0}\left[\lambda\,|\langle T,\varphi\rangle| - \frac{\lambda^q}{q}\int_\Omega |\nabla\varphi|^q\,dx\right].$$

On the other hand, the supremum on the right is readily computed: this corresponds to the choice

$$\lambda = |\langle T,\varphi\rangle|^{\frac{1}{q-1}}\left(\int_\Omega |\nabla\varphi|^q\,dx\right)^{-\frac{1}{q-1}},$$

which gives

$$\sup_{\lambda\ge 0}\left[\lambda\,|\langle T,\varphi\rangle| - \frac{\lambda^q}{q}\int_\Omega |\nabla u|^q\,dx\right] = \frac{1}{p}\left(\frac{|\langle T,\varphi\rangle|}{\|\varphi\|_{\dot W^{1,q}}}\right)^p.$$

Passing to the supremum over $\varphi \in \dot W^{1,q}(\Omega)$ and using the definition of the dual norm, we get the thesis. □

We also denote by $\mathcal{E}'_1(\Omega)$ the space of distributions of order 1, with (compact) support in Ω. In what follows, we will tacitly identify this space with the dual space of the Banach space $C^1(\Omega)$, endowed with the norm

$$\|\varphi\|_{C^1(\Omega)} = \|\varphi\|_{L^\infty(\Omega)} + \|\nabla\varphi\|_{L^\infty(\Omega)}.$$

By denoting ν_Ω the outer normal versor to $\partial\Omega$, we have the following characterization for the dual space $\dot W^{-1,p}(\Omega)$.

Lemma 2.2. *Let $p = q' = q/(q-1)$. Given a vector field $V \in L^p(\Omega;\mathbb{R}^N)$ and $T \in \mathcal{E}'_1(\Omega)$, we say that V satisfies*

$$-\operatorname{div} V = T \quad \text{in } \Omega, \qquad V\cdot\nu_\Omega = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

if

$$\int_\Omega \nabla\varphi(x)\cdot V(x)\,dx = \langle T,\varphi\rangle, \qquad \text{for every } \varphi\in C^1(\Omega).$$

If we set

$$\mathcal{E}'_{1,p}(\Omega) = \{T\in\mathcal{E}'_1(\Omega)\,:\, \text{ there exists } V\in L^p(\Omega;\mathbb{R}^N) \text{ satisfying (2.1)}\},$$

we then have the identification

$$\dot W^{-1,p}(\Omega) = \mathcal{E}'_{1,p}(\Omega).$$

Proof. Let $T\in \dot W^{-1,p}(\Omega)$, first of all we observe that then $T\in\mathcal{E}'_1(\Omega)$ as well. Now, consider the following maximization problem

$$\sup_{v\in\dot W^{1,q}(\Omega)} \langle T, v\rangle - \frac{1}{q}\int_\Omega |\nabla v(x)|^q\,dx.$$

By means of the Direct Methods, it is not difficult to see that there exists a (unique) maximizer $u \in \dot W^{1,q}(\Omega)$ for this problem. Moreover, such a maximizer satisfies the relevant Euler-Lagrange equation, given by

$$\int_\Omega |\nabla u(x)|^{q-2}\, \nabla u(x) \cdot \nabla \varphi(x)\, dx = \langle T, \varphi\rangle, \qquad \text{for every } \varphi \in \dot W^{1,q}(\Omega).$$

By defining $V = |\nabla u|^{q-2}\, \nabla u \in L^p(\Omega;\mathbb{R}^N)$, the previous identity implies $T \in \mathcal{E}'_{1,p}(\Omega)$.

Conversely, let us take $T \in \mathcal{E}'_{1,p}(\Omega)$, then for every $\varphi \in C^1(\Omega)$ equation (2.1) implies

$$|\langle T, \varphi\rangle| = \left|\int_\Omega \nabla\varphi(x) \cdot V(x)\, dx\right| \le \|\varphi\|_{\dot W^{1,q}}\, \|V\|_{L^p(\Omega)}.$$

Using the density of $C^1(\Omega)$ in $\dot W^{1,q}(\Omega)$, we then get that T can be extended in a unique way as an element (that we still denote T, for simplicity) of $\dot W^{-1,q}(\Omega)$. Observe that this extension satisfies

$$\|T\|_{\dot W^{-1,p}(\Omega)} \le \|V\|_{L^p(\Omega)},$$

by taking the supremum in the previous inequality. □

Remark 2.3. We remark that the elements of $\mathcal{E}'_{1,p}(\Omega)$ have "zero average", i.e.

$$\langle T, 1\rangle = 0,$$

as follows by testing the weak formulation of (2.1) with $\varphi \equiv 1$. This is coherent with the previous identification $\dot W^{-1,p}(\Omega) = \mathcal{E}'_{1,p}(\Omega)$, since by construction the space $\dot W^{1,q}(\Omega)$ does not contain any non trivial constant function.

Example 2.4. Consider the measure $T = \delta_a - \delta_b$ for two points $a \ne b \in \mathbb{R}^N$. We claim that

$$T = \delta_a - \delta_b \in \dot W^{-1,p}(\Omega) \qquad \text{if and only if} \qquad 1 \le p < N/(N-1),$$

where Ω is a sufficiently large ball containing a, b in its interior. We will prove this by using the characterization of Lemma 2.2.

Suppose indeed that there exists some $V \in L^p(\Omega)$, such that $-\operatorname{div} V = T$. We pick a ball $B_r(a)$ centered at a and having radius r such that $2\,r < |a-b|$. Then given $\varepsilon < r$, we take a $C^1_0(B_r(a))$ function η_ε such that

$$\eta_\varepsilon \equiv 1 \quad \text{in } B_{r-\varepsilon}(a) \qquad \text{and} \qquad \|\nabla \eta_\varepsilon\|_{L^\infty} \le C\, \varepsilon^{-1}.$$

Thanks to our assumption, we will have

$$1 = \langle T, \eta_\varepsilon\rangle = \int_{B_r(a)} V \cdot \nabla \eta_\varepsilon\, dx,$$

so that

$$\int_{B_r(a)\setminus B_{r-\varepsilon}(a)} |V|\, dx \ge \frac{\varepsilon}{C}.$$

By Hölder inequality, this easily implies a lower bound on the L^p norm of V, namely

$$\int_{B_r(a)} |V|^p\,dx \ge \varepsilon^p\,|B_r(a)\setminus B_{r-\varepsilon}(a)|^{1-p} = C_{N,p}\,\varepsilon^p\,r^{N\,(1-p)}\,\left[1-\left(1-\frac{\varepsilon}{r}\right)^N\right]^{1-p}.$$

We now make the choice $\varepsilon = r/2$, so that from the previous we can infer

$$\int_{B_r(a)} |V|^p\,dx \ge \widetilde{C}_{N,p}\,r^{p+N\,(1-p)} = \widetilde{C}_{N,p}\,r^{N-p\,(N-1)}.$$

The previous estimate clearly contradicts the assumption $V \in L^p(\Omega)$, if the exponent $N-p\,(N-1)$ is not strictly positive. Therefore we see by Lemma 2.2 that $p < N/(N-1)$ is a necessary condition for $T \in \dot W^{-1,p}(\Omega)$.

This condition on p is also sufficient for T to belong to $\dot W^{-1,p}(\Omega)$, as we will now show. Let us set $2\,\tau = |a-b|$, for simplicity let us suppose that $a = (-\tau, 0, \dots, 0)$ and $b = (\tau, 0, \dots, 0)$. Then, using the notation $x = (x_1, x')$ for a generic point in $\mathbb{R}^N$, where $x' \in \mathbb{R}^{N-1}$, we consider the following vector field

$$V_{a,b}(x) = \begin{cases} \dfrac{(x_1+\tau, x')}{(x_1+\tau)^N}, & \text{if } |x'|\le\tau \text{ and } |x'|-\tau \le x_1 \le 0,\\[2ex] \dfrac{(x_1-\tau, x')}{(x_1-\tau)^N}, & \text{if } |x'|\le\tau \text{ and } \tau-|x'| \ge x_1 \ge 0,\\[2ex] (0,\dots,0), & \text{otherwise.}\end{cases}$$

It is easily seen that $\operatorname{div} V_{a,b} = \delta_a - \delta_b$ and that $V_{a,b}$ is supported on the set

$$D_{a,b} = \left\{(x_1, x') \in \mathbb{R}^N\,:\, \frac{|a-b|}{2} \ge |x'| + |x_1|\right\},$$

which is just the the union of two cones centered at a and b, having opening 1 and height $\tau = |a-b|/2$. Also, by construction we have

$$\begin{aligned}\int_{Q_{a,b}} |V_{a,b}(x)|^p\,dx &= 2\int_{-\tau}^0 \int_{\{x'\,:\,|x'|=x_1+\tau\}} \frac{\left(\sqrt{(x_1+\tau)^2+|x'|^2}\right)^p}{(x_1+\tau)^{Np}}\,dx'\,dx_1\\ &= 2^{\frac{p+2}{2}}\,N\,\omega_N \int_{-\tau}^0 (x_1+\tau)^{-Np+p+N-1}\,dx_1\end{aligned}$$

so that finally

$$\|V_{a,b}\|_{L^p}^p \le C_{N,p}\,|a-b|^{N-p(N-1)},$$

thanks to our assumption $p < N/(N-1)$. For some related constructions, the reader is referred to [3, Proposition 3.2] and [32, Lemma 8.3].

As a consequence of Lemma 2.2, we have the following well-posedness result for Beckmann problem.

Proposition 2.5. *Let $\mathcal{H} : \Omega \times \mathbb{R}^N$ be a Carathéodory function, such that $z \mapsto \mathcal{H}(x, z)$ is convex on $\mathbb{R}^N$, for every $x \in \Omega$. We further suppose that $\mathcal{H}$ satisfies the growth conditions*

$$\lambda(|z|^p - 1) \le \mathcal{H}(x, z) \le \frac{1}{\lambda}(|z|^p + 1), \qquad (x, z) \in \Omega \times \mathbb{R}^N, \tag{2.2}$$

for some $0 < \lambda \le 1$. Then the following problem

$$\min_{V \in L^p(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega \mathcal{H}(x, V)\, dx \,:\, -\operatorname{div} V = T, \quad V \cdot \nu_\Omega = 0 \right\}, \tag{2.3}$$

admits a minimizer with finite energy if and only if $T \in \dot{W}^{-1,p}(\Omega)$.

Proof. Let $T \in \dot{W}^{-1,p}(\Omega)$, then thanks to Lemma 2.2 there exists at least one admissible vector field V_0 with finite energy, so that the infimum (2.3) is finite. If $\{V_n\}_{n\in\mathbb{N}} \subset L^p(\Omega; \mathbb{R}^N)$ is a minimizing sequence, then the hypothesis on $\mathcal{H}$ guarantees that this sequence is weakly convergent to some $V \in L^p(\Omega; \mathbb{R}^N)$. Thanks to the convexity of $\mathcal{H}$, the functional is weakly lower semicontinuous, i.e.

$$\begin{aligned}\int_\Omega \mathcal{H}(x, V)\, dx &\le \liminf_{n\to\infty} \int_\Omega \mathcal{H}(x, V_n)\, dx \\ &= \min_{V \in L^p(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega \mathcal{H}(x, V(x))\, dx \,:\, \begin{array}{c} -\operatorname{div} V = T, \\ V \cdot \nu_\Omega = 0 \end{array} \right\}.\end{aligned}$$

Moreover, this limit vector field V is still admissible, since

$$\int_\Omega \nabla\varphi \cdot V\, dx = \lim_{n\to\infty} \int_\Omega \nabla\varphi \cdot V_n\, dx = \langle T, \varphi\rangle, \qquad \text{for every } \varphi \in C^1(\Omega),$$

by weak convergence. Therefore V realizes the minimum.

On the other hand, suppose that $T \notin \dot{W}^{-1,p}(\Omega)$. Again thanks to Lemma 2.2 we have that the set of admissible vector fields is empty, so the problem is not well-posed. □

For the sequel, we need the following definition.

Definition 2.6. We say that a vector field $V \in L^1_{loc}(\Omega; \mathbb{R}^N)$ is *acyclic* if, whenever we can write $V = V_1 + V_2$, with $|V| = |V_1| + |V_2|$ and $\operatorname{div} V_1 = 0$ in distributional sense, there must result $V_1 \equiv 0$.

The following is a mild regularity result for optimizers of (2.3) in the *isotropic* case, i.e. when $\mathcal{H}$ depends on the variable z only through its modulus. This will be crucial in order to equivalently reformulate (2.3) as a Lagrangian problem, where the transport is described by measures on paths.

Proposition 2.7. *Let us suppose that $\mathcal{H}$ satisfies the hypotheses of Proposition 2.5. In addition, we suppose that*

$$z \mapsto \mathcal{H}(x, z) \quad \textit{is a strictly convex increasing function of } |z|, \textit{ for every } x.$$

Then there exists a unique minimizer V for (2.3) and V is acyclic.

Proof. The uniqueness of V just follows by strict convexity, so let us prove that V is acylic. Let us suppose that we can write $V = V_1 + V_2$, for some vector fields $V_1, V_2 \in L^1(\Omega;\mathbb{R}^N)$ such that

$$|V| = |V_1| + |V_2| \qquad \text{and} \qquad \operatorname{div} V_1 = 0.$$

As a consequence, we have $\operatorname{div} V = \operatorname{div} V_2$ and $|V| \geq |V_2|$. Thus V_2 is a competitor for problem (2.3) with energy not larger than that of V, thanks to the monotonicity of $\mathcal{H}$. Since V is the unique minimizer, it must have energy equal to that of V_2. Thus $|V| = |V_2|$ and $|V_1| = 0$ almost everywhere. Owing to the definition, this shows that V is acylic, thus concluding the proof. □

3. Duality for Beckmann problem

In what follows, we will need the following general convex duality result (for the proof, the reader is referred to [17, Proposition 5, page 89]). The statement has been slightly simplified, in order to be directly adapted to our setting.

Convex duality. *Let $\mathcal{F} : Y \to \mathbb{R}$ be a convex lower semicontinuous functional on the reflexive Banach space Y. Let X be another reflexive Banach space and $A : X \to Y$ a bounded linear operator, with adjoint operator $A^* : Y^* \to X^*$. Then we have*

$$\sup_{x\in X} \langle x^*, x\rangle - \mathcal{F}(A\,x) = \min_{y^*\in Y^*} \{\mathcal{F}^*(y^*)\,:\, A^* y^* = x^*\}, \qquad x^* \in X^*, \tag{3.1}$$

where $\mathcal{F}^* : Y^* \to \mathbb{R}\cup\{+\infty\}$ denotes the Legendre-Fenchel transform of $\mathcal{F}$. Moreover, if the supremum in (3.1) *is attained at some $x_0 \in X$, then the minimum in* (3.1) *is attained as well, by a $y_0^* \in Y^*$ such that*

$$y_0^* \in \partial\mathcal{F}(A\,x_0).$$

Thanks to the previous result, we obtain that Beckmann problem admits a dual formulation, which is a classical elliptic problem in Calculus of Variations.

Theorem 3.1 (Duality)**.** *Let $\mathcal{H}$ be a function satisfying the hypotheses of Proposition 2.5 and $T \in \dot W^{-1,p}(\Omega)$. Then*

$$\begin{aligned} &\min_{V\in L^p(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega \mathcal{H}(x, V(x))\,dx \,:\, -\operatorname{div} V = T, \quad V\cdot\nu_\Omega = 0\right\} \\ &\qquad\qquad = \max_{v\in \dot W^{1,q}(\Omega)} \left\{ \langle T, v\rangle - \int_\Omega \mathcal{H}^*(x, \nabla v(x))\,dx \right\}, \end{aligned} \tag{3.2}$$

where $\mathcal{H}^*$ is the partial Legendre-Fenchel transform of $\mathcal{H}$, i.e.

$$\mathcal{H}^*(x,\xi) = \sup_{z\in\mathbb{R}^N} \xi\cdot z - \mathcal{H}(x,z), \qquad x\in\Omega, \xi\in\mathbb{R}^N.$$

Moreover, if $V_0 \in L^p(\Omega)$ and $v_0 \in \dot W^{1,q}(\Omega)$ are two optimizers for the problems in (3.2)*, we have the following primal-dual optimality condition*

$$V_0 \in \partial\mathcal{H}^*(x, \nabla v_0) \qquad \text{in } \Omega, \tag{3.3}$$

where $\partial\mathcal{H}^*$ denotes the subgradient with respect to the ξ variable, i.e.

$$\partial\mathcal{H}^*(x,\xi)=\{z\in\mathbb{R}^N\,:\,\mathcal{H}^*(x,\xi)+\mathcal{H}(x,z)=\xi\cdot z,\quad \textit{for every } x\in\Omega\}.$$

Proof. To prove (3.2), it is sufficient to apply the previous result with the choices

$$Y=L^q(\Omega;\mathbb{R}^N),\quad X=\dot W^{1,q}(\Omega),\quad \mathcal{F}(\phi)=\int_\Omega \mathcal{H}^*(x,\phi(x))\,dx\quad\text{and}\quad A(\varphi)=\nabla\varphi.$$

Observe that the operator A is bounded, since

$$\|A(\varphi)\|_Y=\|\nabla\varphi\|_{L^q(\Omega)}=\|\varphi\|_X,\qquad\text{for every } \varphi\in X,$$

and that

$$\mathcal{F}^*(\xi)=\int_\Omega \mathcal{H}^{**}(x,\xi(x))\,dx=\int_\Omega \mathcal{H}(x,\xi(x))\,dx,$$

since $\xi\mapsto\mathcal{H}(x,\xi)$ is convex and lower semicontinuous, for every $x\in\Omega$. We only need to compute the adjoint operator $A^*:L^p(\Omega;\mathbb{R}^N)\to\dot W^{-1,p}(\Omega)$. Let us define the map $D:L^p(\Omega;\mathbb{R}^N)\to\mathcal{E}'_{1,p}(\Omega)$ by

$$DV\in\mathcal{E}'_1(\Omega)\qquad\text{such that } \langle DV,\varphi\rangle=\int_\Omega \nabla\varphi\cdot V\,dx,\qquad\text{for every } \varphi\in C^1(\Omega).$$

Observe that this is a linear operator, whose image is contained in $\mathcal{E}_{1,p}(\Omega)=\dot W^{-1,p}(\Omega)$ just by construction and by definition of $\mathcal{E}_{1,p}(\Omega)$. Moreover, for every $\varphi\in C^1(\Omega)$ and $V\in L^p(\Omega;\mathbb{R}^N)$ we have

$$\langle A\,\varphi,V\rangle=\int_\Omega \nabla\varphi(x)\cdot V(x)\,dx=\langle\varphi,D\,V\rangle.$$

By density of $C^1(\Omega)$ in $W^{1,q}(\Omega)$, we obtain that $D=A^*$, then (3.2) follows from (3.1).

The primal-dual optimality condition (3.3) as well is a direct consequence of the second part of the convex duality result. It is sufficient to observe that the maximum in (3.2) is attained at some $v_0\in\dot W^{1,p}(\Omega)$, simply using the Direct Methods. This implies that a minimizer V_0 of Beckmann problem has to satisfy

$$V_0\in\partial\mathcal{F}(\nabla v_0),$$

which implies directly (3.3). □

A significant instance of the previous result corresponds to $\mathcal{H}(x,z)=|z|^p$. Thanks to Lemma 2.1, we have the following.

Corollary 3.2. *For every $T\in\dot W^{-1,p}(\Omega)$, we have*

$$\|T\|_{\dot W^{-1,p}(\Omega)}=\min_{V\in L^p(\Omega;\mathbb{R}^N)}\Big\{\|V\|_{L^p(\Omega)}\,:\,-\mathrm{div}\,V=T,\quad V\cdot\nu_\Omega=0\Big\}.$$

Proof. It is sufficient to use (3.2) and Lemma 2.1, paying attention to the easy fact

$$\max_{\varphi\in\dot W^{1,q}(\Omega)}|\langle T,\varphi\rangle|-\frac{1}{q}\int_\Omega|\nabla\varphi|^q\,dx=\max_{\varphi\in\dot W^{1,q}(\Omega)}\langle T,\varphi\rangle-\frac{1}{q}\int_\Omega|\nabla\varphi|^q\,dx.$$

This establishes the thesis. □

Corollary 3.3. *Under the hypotheses of Theorem 3.1, we have that the functional*

$$\begin{array}{cccc} \mathfrak{F}_{\mathcal{H}}: & \dot{W}^{-1,p}(\Omega) & \to & \mathbb{R}^+ \\ & T & \mapsto & \textit{minimal value } (2.3) \end{array}$$

is convex and weakly lower semicontinuous.

Proof. It is sufficient to observe that thanks to Theorem 3.1, the value (2.3) can be written as a supremum of the linear functionals L_φ defined by

$$L_\varphi(T) = \langle T, \varphi \rangle - \int_\Omega \mathcal{H}^*(x, \nabla \varphi)\, dx, \qquad \varphi \in \dot{W}^{1,q}(\Omega).$$

Then the thesis follows. □

About the duality result of Theorem 3.1, some comments are in order.

Remark 3.4 (Economic interpretation)**.** By the so-called *Legendre reciprocity formula* in Convex Analysis, the primal-dual optimality condition (3.3) can be equivalently written as

$$\nabla v_0 \in \partial \mathcal{H}(x, V_0), \qquad \text{in } \Omega, \tag{3.4}$$

so this result is the rigorous justification of the necessary optimality conditions derived in [2, Lemma 2]. Such a function v_0 will be called a *Beckmann potential* and its economic interpretation is that of an efficiency price system. It can be seen as a generalization of a Kantorovich potential, to a situation where the cost to move some unit of mass from x to y is not fixed, but it depends on the quantity of traffic generated by the transport V_0 itself. Heuristically, observe that in such a situation this minimal cost will be given by the "congested metric"

$$d_{V_0}(x_0, x_1) = \min_{\gamma\,:\,\gamma(i)=x_i} \int_0^1 \left|\nabla \mathcal{H}(\cdot, V_0) \circ \gamma\right| \left|\gamma'(t)\right|\, dt.$$

This means that each mass particle is charged for the marginal cost it produces, the latter being the derivative of the function $\mathcal{H}$ (we suppose for simplicity that $\mathcal{H}$ has a true gradient and not just a subgradient). Then v_0 acts as a Kantorovich potential for the Optimal Transport problem

$$\min \left\{ \int_{\Omega\times\Omega} d_{V_0}(x,y)\, d\eta(x,y) \,:\, (\pi_x)_\# \eta = T_+ \quad \text{and} \quad (\pi_y)_\# \eta = T_- \right\},$$

where we suppose for simplicity that $T = T_+ - T_-$, with T_+ and T_- positive measures having the same mass. It should be remarked that $\nabla \varphi_0$ does not give the direction of optimal transportation in Beckmann problem, since $\nabla \varphi_0$ and V_0 are linked through the relation (3.4), thus they are not parallel in general. Actually, this is the case only when the cost function $\mathcal{H}$ is *isotropic*, i.e. when it just depends on $|V|$, for every admissible vector field V. This is the case studied by Beckmann in his original paper [2].

Remark 3.5 (Regularity of optimal vector fields)**.** We point out that if $z \mapsto \mathcal{H}(x,z)$ is strictly convex, then $\xi \mapsto \mathcal{H}^*(x,\xi)$ is C^1. Thus in this case, the optimal V_0 is unique and we have

$$V_0 = \nabla \mathcal{H}^*(x, \nabla v_0).$$

Then the regularity of the optimal vector field V_0 can be recovered from the regularity of a Beckmann potential, which solves the following elliptic boundary value problem

$$\left\{ \begin{array}{rcll} -\mathrm{div}\, \nabla \mathcal{H}^*(x, \nabla u) & = & T, & \text{in } \Omega \\ \nabla \mathcal{H}^*(x, \nabla u) \cdot \nu_\Omega & = & 0, & \text{on } \partial\Omega. \end{array} \right. \tag{3.5}$$

For instance, if $\mathcal{H}^*$ in uniformly convex "at infinity", i.e if there exists $M > 0$ such that

$$(1 + |z|^2)^{\frac{q-2}{2}} \leq C \min_{|\xi|=1} \langle D^2\mathcal{H}^*(x, z)\, \xi, \xi\rangle, \qquad \text{for every } |z| \geq M,\ x \in \Omega,$$

and

$$|D^2\mathcal{H}^*(x, z)| \lesssim (1 + |z|^2)^{\frac{q-2}{2}}, \qquad (x, z) \in \Omega \times \mathbb{R}^N,$$

then V is bounded, provided that $T \in L^{N+\varepsilon}(\Omega)$, since in this case solutions to (3.5) are Lipschitz. These assumptions are verified for example by (see [10])

$$\mathcal{H}^*(z) = \frac{1}{q}(|z| - \delta)_+^q, \qquad z \in \mathbb{R}^N,$$

where $(\cdot)_+$ stands for the positive part and $\delta \geq 0$, but are violated by anisotropic functions of the type

$$\mathcal{H}^*(z) = \sum_{i=1}^N \frac{1}{q}\,(|z_i| - \delta_i)_+^q, \qquad z \in \mathbb{R}^N,$$

considered for example in [8, 9].

4. A Lagrangian reformulation

The aim of this section is to introduce a Lagrangian counterpart of Beckmann model and to show how the two models turn out to be equivalent. The model we are going to present is a continuous version of the classical discrete model by Wardrop, which we mentioned above. This continuous model has already been introduced in [12] and the equivalence has been discussed in [10]. Here, paralleling the case of Beckmann model, we go one step further, by extending this model to its natural setting, i.e. the dual space $\dot{W}^{-1,p}(\Omega)$. Moreover, we will prove equivalence of the models without imposing any additional regularity on the datum T. This will be achieved by means of Smirnov decomposition theorem for $1-$currents (see Theorem A.20).

Given two Lipschitz curves $\gamma_1, \gamma_2 : [0, 1] \to \Omega$, we say that they are *equivalent* if there exists a continuous surjective nondecreasing function $\mathfrak{t} : [0, 1] \to [0, 1]$ such that

$$\gamma_2(t) = \gamma_1(\mathfrak{t}(t)), \qquad \text{for every } t \in [0, 1].$$

Let us call $\mathcal{L}(\Omega)$ the set of all equivalence classes of Lipschitz paths in Ω. We can introduce a topology on this set by defining the following distance

$$d(\gamma_1, \gamma_2) := \max \left\{|\hat{\gamma}_1(t) - \hat{\gamma}_2(t)| :\ t \in [0, 1],\ \hat{\gamma}_i \text{ equivalent to } \gamma_i\right\}.$$

Observe that convergence in this metric is nothing but the usual uniform convergence, up to reparametrizations.

We denote the class of σ-finite positive Borel (with respect to the above topology) measures on $\mathcal{L}(\Omega)$ by $\mathscr{M}_+(\mathcal{L}(\Omega))$. For $Q \in \mathscr{M}_+(\mathcal{L}(\Omega))$, we may define the corresponding *traffic intensity* by

$$\langle i_Q, \varphi\rangle := \int_{\mathcal{L}(\Omega)} \left(\int_0^1 \varphi(\gamma(t))\, |\gamma'(t)|\, dt\right) dQ(\gamma), \qquad \varphi \in C(\Omega),$$

provided that the outer integrals converges, in which case we say that "the traffic intensity i_Q exists". If this is the case, then the following integral

$$\langle \mathbf{i}_Q, \varphi\rangle = \int_{\mathcal{L}(\Omega)} \left(\int_0^1 \varphi(\gamma(t)) \cdot \gamma'(t)\, dt\right) dQ(\gamma), \qquad \varphi \in C(\Omega; \mathbb{R}^N).$$

also converges. These definitions do not depend on the particular representative of the equivalence class we chosen, since the integrals in brackets are invariant under time reparameterization.

Remark 4.1. Observe that i_Q counts in a scalar way the traffic generated by Q, while $\mathbf{i}_Q$ computes it in a vectorial way. This means that in principle i_Q and $|\mathbf{i}_Q|$ could be very different, since in $\mathbf{i}_Q$ two huge amounts of mass going in opposite direction give rise to a lot of cancellations, as the orientation of curves is taken into account. As a simple example, suppose to have two distinct points $x_0 \neq x_1$, then we consider the measure

$$Q = \frac{1}{2}\,\delta_{\gamma_1} + \frac{1}{2}\,\delta_{\gamma_2},$$

with $\gamma_1(t) = (1-t)\,x_0 + t\,x_1$ and $\sigma_2(t) = (1-t)\,x_1 + t\,x_0$, that is we simply exchange the mass in x_0 with that in x_1 and vice versa. By computing the traffic intensity, we get

$$i_Q = \mathscr{H}^1 \llcorner \overline{x_0 x_1},$$

which takes into account the intuitive fact that on the segment $\overline{x_0 x_1}$ globally there is a non negligible amount of transiting mass. On the other hand, it is easily seen that

$$\mathbf{i}_Q \equiv 0.$$

Given $T \in \mathcal{E}'_1(\Omega)$, we define also the following space of measures

$$\mathcal{Q}_p(T) := \Big\{Q \in \mathscr{M}_+(\mathcal{L}(\Omega)) :\ i_Q \in L^p(\Omega) \ \text{ and } \ (e_0 - e_1)_\# Q = T\Big\},$$

where $e_i : \mathcal{L}(\Omega) \to \Omega$ is defined by $e_i(\gamma) = \gamma(i)$, for $i = 0, 1$ and the equality $(e_0 - e_1)_\# Q = T$ has to be understood in the distributional sense, i.e.

$$\int_{\mathcal{L}(\Omega)} [\varphi(\gamma(0)) - \varphi(\gamma(1))]\, dQ(\gamma) = \langle T, \varphi\rangle, \qquad \text{for every } \varphi \in C^1(\Omega).$$

The convergence of the above integral is again in the sense of distributions.

Remark 4.2. All these definitions are reformulated in Appendix A in terms of currents.

Let us now consider $\mathcal{H} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ a Carathéodory function such that

$$\lambda\,(t^p - 1) \le \mathcal{H}(x,t) \le \frac{1}{\lambda}\,(t^p + 1), \qquad x \in \Omega,\, t \in \mathbb{R}^+ \tag{4.1}$$

for some $0 < \lambda \le 1$ and such that

$$t \mapsto \mathcal{H}(x,t) \ \text{ is convex, for every } x \in \Omega.$$

If $\mathcal{Q}_p(T) \neq \emptyset$, then we may define the following minimization problem:

$$\inf_{Q \in \mathcal{Q}_p(T)} \int_\Omega \mathcal{H}(x, i_Q(x))\,dx. \tag{$\mathcal{M}_\mathcal{H}$}$$

We will show that the above minimization is well-posed and equivalent to the one in (2.3). At this aim, we will use the following result, proved in the appendix (see Theorem A.20) in a slightly stronger formulation in terms of currents.

Proposition 4.3. *Let $1 \le p < \infty$. Suppose that $V \in L^p(\Omega, \mathbb{R}^N)$ and that V is acyclic. If we set $T = -\operatorname{div} V$, then it is possible to find a positive $\sigma-$finite Borel measure $Q \in \mathscr{M}_+(\mathcal{L}(\Omega))$ such that*

$$(e_0 - e_1)_\# Q = T \qquad \text{in distributional sense.}$$

Moreover, we have

$$\mathbf{i}_Q = V \qquad \text{and} \qquad i_Q = |V|.$$

In particular $Q \in \mathcal{Q}_p(T)$.

Thanks to the previous result, we can at first give a necessary and sufficient condition for the set $\mathcal{Q}_p(T)$ to be not empty.

Proposition 4.4. *The set $\mathcal{Q}_p(T)$ is not empty if and only if $T \in \dot W^{-1,p}(\Omega)$.*

Proof. Let us suppose that $T \notin \dot W^{-1,p}(\Omega)$ and assume by contradiction that there exists $Q_0 \in \mathcal{Q}_p(T)$. In particular

$$\int_\Omega |i_{Q_0}(x)|^p\,dx < +\infty. \tag{4.2}$$

The vector measure $\mathbf{i}_{Q_0}$ satisfies (2.1), since

$$\begin{aligned}\int_\Omega \nabla\varphi \cdot d\mathbf{i}_{Q_0} &= \int_{\mathcal{L}(\Omega)} \left(\int_0^1 \nabla\varphi(\gamma(t)) \cdot \gamma'(t)\,dt\right) dQ_0(\gamma)\\ &= \int_{\mathcal{L}(\Omega)} \Big[\varphi(\gamma(1)) - \varphi(\gamma(0))\Big]\,dQ_0(\gamma) = \langle T, \varphi\rangle,\end{aligned}$$

for every $\varphi \in C^1(\Omega)$. Also, thanks to the fact that $|\mathbf{i}_{Q_0}| \le i_{Q_0}$ and to (4.2), we have that $\mathbf{i}_{Q_0} \in L^p(\Omega; \mathbb{R}^N)$. This contradicts the fact that $T \notin \dot W^{-1,p}(\Omega)$, as wanted.

Let us now take $T \in \dot W^{-1,p}(\Omega)$, then there exists a minimizer V of (2.3). Thanks to Proposition 2.7, we know that V is acyclic, so that by Proposition 4.3 we can infer the existence of $Q \in \mathcal{Q}_p(T)$. This gives directly the thesis. □

We can now prove our equivalence result, which is the main result of this section. Observe that we prove at the same time existence of a minimizer for $(\mathcal{M}_{\mathcal{H}})$.

Theorem 4.5. *Let $\mathcal{H} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ be a Carathéodory function satisfying (4.1) and such that*

$$t \mapsto \mathcal{H}(x,t) \quad \text{is convex and increasing}, \qquad x \in \Omega.$$

For every $T \in \dot{W}^{-1,p}(\Omega)$, we have

$$\inf_{Q \in \mathcal{Q}_p(T)} \int_\Omega \mathcal{H}(x, i_Q(x))\, dx = \min_{V \in L^p(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega \mathcal{H}(x,|V|)\, dx \ : \ \begin{array}{c} -\mathrm{div}\, V = T, \\ V \cdot \nu_\Omega = 0 \end{array} \right\}, \tag{4.3}$$

and the infimum on the left-hand side is achieved.

Moreover, if $Q_0 \in \mathcal{Q}_p(T)$ is optimal, then $\mathbf{i}_{Q_0} \in L^p(\Omega;\mathbb{R}^N)$ is a minimizer of Beckmann problem. Conversely, if V_0 is optimal, then there exists $Q_{V_0} \in \mathcal{Q}_p(T)$ such that $i_{Q_{V_0}} = |\mathbf{i}_{Q_{V_0}}|$ minimizes the Lagrangian problem.

Proof. Since $T \in \dot{W}^{-1,p}(\Omega)$, we first observe that by the previous result the set $\mathcal{Q}_p(T)$ is not empty. Moreover, for every admissible Q, we have $|\mathbf{i}_Q| \le i_Q$ and $\mathbf{i}_Q$ is admissible for Beckmann problem. Using the monotonicity of $\mathcal{H}(x,\cdot)$, we then obtain

$$\min_{V \in L^p(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega \mathcal{H}(x,|V|)\, dx \ : \ \begin{array}{c} -\mathrm{div}\, V = T, \\ V \cdot \nu_\Omega = 0 \end{array} \right\} \le \inf_{Q \in \mathcal{Q}_p(T)} \int_\Omega \mathcal{H}(x, i_Q(x))\, dx < +\infty.$$

Let us now take a minimizer $V_0 \in L^p(\Omega;\mathbb{R}^N)$ for Beckmann problem, then by Proposition 2.7 this is acyclic. Thus by Proposition 4.3, there exists $Q_0 \in \mathcal{Q}_p(T)$ such that $|V_0| = i_{Q_0}$, i.e.

$$\min_{V \in L^p(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega \mathcal{H}(x,|V|)\, dx \ : \ \begin{array}{c} -\mathrm{div}\, V = T, \\ V \cdot \nu_\Omega = 0 \end{array} \right\} = \int_\Omega \mathcal{H}(x, i_{Q_0}(x))\, dx.$$

This shows that (4.3) holds true and that the infimum in the left-hand side is indeed a minimum.

The relation between minimizers of the two problems is an easy consequence of the previous constructions. □

Remark 4.6. Similar Lagrangian formulations have been studied in connection with transport problems involving *concave costs*, like for example problems where to move a mass m of a length ℓ costs $m^\alpha\,\ell$ $(0 < \alpha < 1)$. For these the reader is referred to the monograph [4], as well as to the papers [29, 38].

Appendix A. Decompositions of acyclic flat 1-currents

A.1. **Definitions and links to vector fields.** The classical references which we use for currents are [19, 21]. We translate however all results in the language of the previous sections. In the following, by $\Omega \subset \mathbb{R}^N$ we still denote the closure of an open bounded connected set, having smooth boundary.

Definition A.1. A 0-*current* on Ω is a distribution on Ω, in the usual sense. A 1-*current* on Ω is a vector valued distribution on Ω. The relevant duality is the one with 1-forms $\omega(x) = \sum_{i=1}^N \omega_i(x)\, dx^i$ having smooth coefficients, i.e. $\omega_i \in C^\infty(\Omega)$. We denote by $C^\infty(\Omega, \wedge^1\mathbb{R}^N)$ the space of such forms. More generally, a k-current is an element in the dual of smooth k-forms $C^\infty(\Omega, \wedge^k\mathbb{R}^N)$.

The above definition automatically gives the space of currents a natual weak topology, defined via the duality with smooth forms. For any current there is a natural definition of boundary.

Definition A.2. If I is a k-current on Ω, then we can define its *boundary* ∂I to be the $(k-1)$-current on Ω which satisfies

$$\langle \partial I, \varphi\rangle = -\langle I, d\varphi\rangle, \qquad \text{for all } \varphi \in C^\infty(\Omega, \wedge^{k-1}\mathbb{R}^N),$$

where d is the exterior derivative. For example, if $k = 1$ we must take $\varphi \in C^\infty(\Omega)$ and $d\varphi$ is the 1-form $\sum_i \partial_{x_i}\varphi\, dx_i$.

Definition A.3. Let I be a k-current, then its *mass* is

$$\mathbb{M}(I) = \sup\left\{ |\langle I, \omega\rangle| \,:\, \omega \in C^\infty(\Omega, \wedge^k\mathbb{R}^N),\ \sup_{x\in\Omega} \|\omega(x)\| \le 1 \right\},$$

where the norm $\|\omega\|$ for an alternating k-tensor is defined as

$$\|\omega\| = \sup\{\langle \omega, e\rangle \,:\, e \text{ unit simple } k-\text{vector}\}.$$

For $k = 1$ this coincides with the usual norm $\|\omega\| = \sqrt{\omega_1^2 + \ldots + \omega_N^2}$.

We will be interested just in 1-currents which are distributions of order 0 (i.e. vector valued Radon measures) and in their boundaries (which are scalar distributions of order 1). For them, a comment is in order.

Remark A.4. Finite mass 1-currents can be identified with vector-valued Radon measures as follows. To every smooth 1-form ω we may associate naturally a vector field $X_\omega := (\omega_1, \ldots, \omega_N)$. We can then write for a 1-current I

$$\int X_\omega\, dI := \langle I, \omega\rangle.$$

Since C^∞ is dense in C^0, the resulting linear functional on smooth vector fields can be identified via Hahn-Banach theorem to a unique linear functional on C^0 vector fields. The latter is indeed a vector-valued Radon measure by Riesz representation theorem. Since $\|\omega(x)\| = \|X_\omega(x)\|$ by the above definition, we automatically obtain that

$$\mathbb{M}(I) = \int_\Omega d|I|,$$

i.e. the mass equals the total variation of I regarded as a Radon measure. The same reasoning can be applied to 0-currents of finite mass, by identifying them with scalar Radon measures[3].

[3] See also "*Distributions representable by integration*" in [19, 4.1.7]

Definition A.5 (variation of a current)**.** Let A be a k-current with $k \in \{0,1\}$. Then we may define the variation measure μ_A of A in the usual sense, by identifying A with a Radon measure as in Remark A.4. Thus for a Borel set E we define

$$\mu_A(E) := \sup\left\{\sum_{i=1}^{k}\left|\int_{E_i} dA\right| : E_i \text{ form a Borel partition of } E\right\}.$$

An equivalent way of defining μ_A would be as the infimum of all measures μ such that $\langle A,\omega\rangle \le \int_\Omega \|\omega\| d\mu$ for all smooth 1-forms ω.

We recall that a $k-$current T is said to be *normal* if[4]

$$\mathbb{M}(T) + \mathbb{M}(\partial T) < +\infty.$$

We now define *flat currents*, a class useful for its closure properties.

Definition A.6. We define the *flat norm* of a k-current A as follows

$$\mathbb{F}(A) = \inf\left\{\mathbb{M}(A - \partial I) + \mathbb{M}(I) \,:\, I \text{ is a } (k+1)-\text{current with } \mathbb{M}(I) < \infty\right\}.$$

Then the space of *flat k-currents* is defined as the completion of normal $k-$currents in the flat norm.

Flat currents of finite mass have the following characterization, which will be exploited in the sequel.

Lemma A.7. *Let T be a flat $k-$current of finite mass. Then T is a flat current if and only if there exists a sequence of normal $k-$currents $\{T_n\}_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \mathbb{M}(T_n - T) = 0.$$

Proof. This is a standard fact, we provide a proof for the ease of completeness. By definition of flat convergence, there exists a sequence $\{I_n\}_{n\in\mathbb{N}}$ of normal $k-$currents and a sequence $\{Y_n\}_{n\in\mathbb{N}}$ of $(k+1)-$currents such that

$$\lim_{n\to\infty}\left[\mathbb{M}(I - I_n - \partial Y_n) + \mathbb{M}(Y_n) - \frac{1}{n}\right] \le \lim_{n\to\infty} \mathbb{F}(I - I_n) = 0.$$

We then set $T_n = I_n + \partial Y_n$, which by construction is a $k-$current and of course

$$\lim_{n\to\infty} \mathbb{M}(T - T_n) = 0,$$

thanks to the previous. Moreover, this is a normal current, since by triangular inequality we have

$$\mathbb{M}(T_n) \le \mathbb{M}(I_n) + \mathbb{M}(\partial Y_n) \le 2\,\mathbb{M}(I_n) + \mathbb{M}(T - I_n - \partial Y_n) + \mathbb{M}(T) < +\infty,$$

thanks to the fact that T has finite mass, and also

$$\mathbb{M}(\partial T_n) = \mathbb{M}(\partial I_n) < +\infty,$$

since $\partial(\partial Y_n) = 0$. This concludes the proof.

[4]For $k = 0$, we define $\partial A = 0$ and thus the condition on ∂A can be omitted.

The converse implication is even simpler: by definition of flat norm, we directly have

$$\mathbb{F}(T - T_n) \leq \mathbb{M}(T - T_n),$$

which concludes the proof. $\square$

A significant instance of flat $1-$currents with finite mass is given by L^1 vector fields. We provide a small proof of this elementary fact.

Lemma A.8. *Given $V \in L^1(\Omega;\mathbb{R}^N)$, we naturally associate to it the $1-$current I_V of finite mass, defined by*

$$\langle I_V, \omega\rangle = \sum_{i=1}^N \int_\Omega \omega_i\, V_i\, dx := \int_\Omega \omega(V), \qquad \textit{for every}\ \ \omega \in C^\infty(\Omega, \wedge^1\mathbb{R}^N).$$

This has compact support contained in Ω and $\mathbb{M}(I_V) = \|V\|_{L^1}$. Moreover, I_V is a flat current.

Proof. We just prove that I_V is a flat current, the first statement being straightforward. At this aim, we will use the characterization of Lemma A.7 and we will construct the approximanting currents by convolution. For every $\varepsilon \ll 1$, we define

$$\Omega_\varepsilon = \{x \in \Omega\,:\, \mathrm{dist}(x, \partial\Omega) > 2\,\varepsilon\},$$

then we take a standard convolution kernel $\varrho \in C^\infty_0$, supported on the ball $\{x\,:\,|x| \leq 1\}$ and we define

$$\varrho_\varepsilon(x) = \varepsilon^{-N}\,\varrho\left(\frac{x}{\varepsilon}\right), \qquad x \in \mathbb{R}^N.$$

We also set

$$V_\varepsilon := (V \cdot 1_{\Omega_\varepsilon}) * \varrho_\varepsilon,$$

where 1_E stands for the characteristic function of a set E. Consequently, we define $I_\varepsilon := I_{V_\varepsilon}$ and observe that V_ε (and thus I_ε) has compact support contained in Ω. From the mass estimate and Hölder inequality we obtain that masses are equi-bounded, since

$$\mathbb{M}(I_\varepsilon) \leq \|V_\varepsilon\|_{L^1} \leq \|\varrho\|_{L^\infty}\|V\|_{L^1} \leq C\|V\|_{L^1}.$$

The boundedness of ∂I_ε follows a similar route: since V_ε have compact support (strictly contained) in Ω, we have

$$\begin{aligned} |\langle \partial I_\varepsilon, \varphi\rangle| = |\langle I_\varepsilon, d\,\varphi\rangle| &= \left|\int_\Omega V_\varepsilon \cdot \nabla\varphi\, dx\right| \\ &= \left|\int_\Omega \mathrm{div} V_\varepsilon\,\varphi\, dx\right| \leq \|\mathrm{div}\, V_\varepsilon\|_{L^1}\,\|\varphi\|_{L^\infty}, \end{aligned}$$

so that setting $C_\varepsilon = \|\mathrm{div}\, V_\varepsilon\|_{L^1}$ and passing to the supremum on φ, from the previous we obtain

$$\mathbb{M}(\partial I_\varepsilon) \leq C_\varepsilon < +\infty.$$

This implies that $\{I_\varepsilon\}_{\varepsilon>0}$ is a sequence of normal currents. Moreover, the mass convergence $\mathbb{M}(I_\varepsilon - I)$ easily follows from the convergence of V_ε to V in $L^1(\Omega;\mathbb{R}^N)$. $\square$

Remark A.9. With the previous notation, it is easily seen that the boundary ∂I_V corresponds to the distributional divergence of V, i.e.

$$\langle \partial I_V, \varphi\rangle = -\int_\Omega \nabla\varphi\cdot V\,dx \qquad \text{for every } \varphi\in C^\infty(\Omega).$$

This distribution as well has compact support in Ω.

Definition A.10. A 1-current I is called *acyclic* if, whenever we can write $I = I_1 + I_2$, with $\mathbb{M}(I) = \mathbb{M}(I_1) + \mathbb{M}(I_2)$ and $\partial I_1 = 0$, there must result $I_1 = 0$.

For $I = I_V$ with $V \in L^1$, we have the correspondence with Definition 2.6, i.e.

$$V \text{ is acyclic} \Longleftrightarrow I_V \text{ is acyclic.} \tag{A.1}$$

A.2. **Lipschitz curves as currents.** We recall that we called $\mathcal{L}(\Omega)$ the space of equivalence classes of Lipschitz curves $\gamma : [0,1] \to \Omega$ (see the beginning of Section 4), with the topology of uniform convergence.
Here we remark that to each $\gamma \in \mathcal{L}(\Omega)$ we may associate a vector valued distribution, i.e. a 1-current, denoted by $[\gamma]$ and defined by requiring for all $\omega \in C^\infty(\Omega, \wedge^1\mathbb{R}^N)$

$$\langle[\gamma],\omega\rangle := \int_\gamma \omega = \int_0^1 \omega(\gamma(t))\,[\gamma'(t)]\,dt = \sum_{i=1}^N \int_0^1 \omega_i(\gamma(t))\,\gamma_i'(t)\,dt.$$

Note that this expression is well-defined on $\mathcal{L}(\Omega)$, since the integral on the right is invariant under reparameterization.

For $\gamma \in \mathcal{L}(\Omega)$ we have $\mathbb{M}([\gamma]) \le \ell(\gamma) := \int_0^1 |\gamma'(t)|\,dt$ with equality exactly when γ has a representative which is injective for $\mathcal{H}^1$-almost every time. We are interested in a stronger requirement, namely that the curve does not even intersect itself, so that an injective representative exists (such curves are called "arcs" in [27]). We fix a notation for such classes of curves.

Definition A.11 (Arcs)**.** We define $\widetilde{\mathcal{L}}(\Omega)$ the subset of $\mathcal{L}(\Omega)$ made of those classes of curves $\gamma : [0,1] \to \Omega$ which have an injective representative.

Remark A.12. One could think of the above-defined arcs as "acyclic curves", where a "cycle" can mean two things:

- we can have a cycle in the parameterization, where a cycle would be represented by a curve satisfying (up to reparameterization) $\gamma(t) = \gamma(1-t)$ and "inserting a cycle of type γ" in another curve $\overline{\gamma}$ such that $\overline{\gamma}(t_0) = \gamma(0)$ would result into the curve:

$$\widetilde{\gamma}(s) = \begin{cases} \overline{\gamma}(2s) & \text{if } s\in[0,t_0/2] \\ \gamma(2s-t_0) & \text{if } s\in[t_0/2,(t_0+1)/2] \\ \overline{\gamma}(2s-1) & \text{if } s\in[(t_0+1)/2,1]. \end{cases}$$

- a curve which intersects itself (i.e. which has no injective parameterization) will instead give rise to a 1-current which is not acyclic, since it will have a reparameterization containing an injectively parameterized loop.

The fact that in the decomposition of acyclic currents one restricts to using just arcs (for which neither type of cycle occurs) is then another natural consequence of the robustness of the acyclicity requirement.

On 1-currents we consider the topology of distributions. The following result links the two topologies:

Lemma A.13. *The map $\mathcal{L}(\Omega) \ni \gamma \mapsto [\gamma]$ as defined above is continuous on the sublevels of the lenght functional $\ell : \mathcal{L}(\Omega) \to \mathbb{R}$.*

Proof. Assuming that $\gamma_i \to \gamma$ and $\ell(\gamma_i) \leq C$ we then obtain that γ_i converge uniformly. In particular they converge as distributions. □

A.3. **Smirnov decomposition theorem.** We are now in a position of stating the theorem on the decomposition of 1-currents due to Smirnov [36] and recenlty extended by Paolini and Stepanov in [27, 28] to metric spaces. The result which is relevant for us is contained in [36], though for its extension needed below we will use the techniques of [27].

Theorem A.14 ([27, 36])**.** *Suppose that I is a normal acyclic 1-current on Ω. Then there exists a finite Radon measure Q on the space of continuous curves on Ω, concentrated on $\widetilde{\mathcal{L}}(\Omega)$, and such that the following decompositions of I are valid in the sense of distributions:*

$$I = \int_{\mathcal{L}(\Omega)} [\gamma] \, dQ(\gamma) \qquad \textit{and} \qquad \mu_I = \int_{\mathcal{L}(\Omega)} \mu_{[\gamma]} \, dQ(\gamma); \tag{A.2}$$

and

$$\partial I = \int_{\mathcal{L}(\Omega)} \partial[\gamma] \, dQ(\gamma) \qquad \textit{and} \qquad \mu_{\partial I} = \int_{\mathcal{L}(\Omega)} \mu_{\partial[\gamma]} \, dQ(\gamma). \tag{A.3}$$

We now note down some reformulations of the items present in the above theorem, in terms of measures and vector fields:

- the total variation is the mass norm, i.e. $\mu_I(\Omega) = \mathbb{M}(I)$ and $\mu_{\partial I}(\Omega) = \mathbb{M}(\partial I)$;
- if V is a L^1_{loc} vector field, then I_V has variation measure $\mu_{I_V} = |V| \cdot \mathscr{L}^N$;
- if $\rho = \rho_+ - \rho_-$ is the decomposition of a signed Radon measure into positive and negative part, then $\mu_\rho = \rho_+ + \rho_-$;
- in particular for $\gamma \in \mathcal{L}(\Omega)$ we have $\mu_{\partial[\gamma]} = \delta_{\gamma(1)} + \delta_{\gamma(0)}$. Since this measure has total variation 2 for all γ, we can quantify the total mass of the above Q by means of the mass norm of the boundary of I. Namely, we have
$$Q(\mathcal{L}(\Omega)) = \frac{1}{2} \, \mu_{\partial I}(\Omega) = \frac{\mathbb{M}(\partial I)}{2};$$
- for $\gamma \in \widetilde{\mathcal{L}}(\Omega)$, there holds $\mu_{[\gamma]} = \mathscr{H}^1 \llcorner \mathrm{Im}(\gamma)$ is the arclenght measure of γ;
- by expanding the definitions and comparing to Section 4, we see that for $I = I_V$ with $V \in L^1(\Omega)$ and for Q as in Theorem A.14, there holds
$$|V| \cdot \mathscr{L}^N = \mu_{I_V} = i_Q \cdot \mathscr{L}^N, \qquad V \cdot \mathscr{L}^N = I = \mathbf{i}_Q \cdot \mathscr{L}^N$$

and

$$\operatorname{div} V = \partial I_V = (e_0 - e_1)_\# Q.$$

All these reformulations allow to translate Theorem A.14 in the case of $I = I_V$, with $V \in L^p(\Omega)$, for $p \geq 1$. This is the content of the next result. Observe that since for a bounded Ω we have the continuous inclusion $L^p(\Omega) \hookrightarrow L^1(\Omega)$, we can restrict the result to $p = 1$ without loss of generality.

Corollary A.15 (Reformulation of Theorem A.14)**.** *Suppose that $V \in L^1(\Omega)$ is an acyclic vector field such that $\operatorname{div} V$ is a signed Radon measure. Then there exists a finite Radon measure Q on the space of continuous curves on Ω, concentrated on $\widetilde{\mathcal{L}}(\Omega)$, and such that the following decompositions of V are valid in the sense of distributions:*

$$\mathbf{i}_Q = V \qquad \textit{and} \qquad i_Q = |V|, \tag{A.4}$$

and

$$-\operatorname{div} V = (e_1 - e_0)_\# Q \qquad \textit{and} \qquad (\operatorname{div} V)_+ + (\operatorname{div} V)_- = (e_1 + e_0)_\# Q. \tag{A.5}$$

We are now lead to consider the extension of Theorem A.14 to the case where $\operatorname{div} V$ is not a Radon measure. This is necessary since in our problem $\operatorname{div} V \in \dot{W}^{-1,p}(\Omega)$, so in general it is not a measure. Some non trivial issues arise in this case: first of all, the measure Q which decomposes I may not be finite, in general.

Example A.16. For $p < \frac{N}{N-1}$, we consider an infinite sequence of small dipoles $\{(a_i, b_i)\}_i$ such that

$$\sum_{i=1}^{\infty} |a_i - b_i|^{N-p(N-1)} < +\infty, \qquad \text{and } D_{a_i,b_i} \text{ are disjoint},$$

where as in Example 2.4 we used the notation

$$D_{a_i,b_i} = \left\{ (x_1, x') \in \mathbb{R}^N \,:\, |x'| + |x_1| \leq \frac{|a_i - b_i|}{2} \right\}.$$

If we consider the vector fields V_{a_i,b_i} as in Example 2.4, then the new vector field defined by $V = \sum_{i=1}^{\infty} V_{a_i,b_i}$ verifies

$$\|V\|^p_{L^p} = \left\| \sum_{i=1}^{\infty} V_{a_i,b_i} \right\|^p_{L^p} \leq C_N \sum_{i=1}^{\infty} |a_i - b_i|^{N-p\,(N-1)} < +\infty,$$

which implies $T := \sum_i (\delta_{a_i} - \delta_{b_i}) \in \dot{W}^{-1,p}(\Omega)$. By observing that $\infty = \mathbb{M}(T) = \int_{\mathcal{L}(\Omega)} dQ$ for any decomposing measure, we see that no finite measure Q can be found. On the other hand, a $\sigma-$finite measure Q can be found, since each V_{a_i,b_i} can be separately decomposed with a measure Q_i of mass 2 and the Q_i have disjoint supports.

Example A.17. The following is another version of Example A.16, which exploits the Sobolev embedding theorem. Let us take $q > N$, then $\dot{W}^{1,q}(\Omega)$ can be identified with a

space of functions which are Hölder continuous of exponent $\alpha = 1 - N/q$. Then we pick the following two curves (here $\varepsilon \ll 1$)

$$\gamma_1(t) = \frac{1}{t^{2/\alpha}}\,(\cos t, \sin t) \qquad \text{and} \qquad \gamma_2(t) = \frac{1-\varepsilon}{t^{2/\alpha}}\,(\cos t, \sin t), \qquad t \geq 1.$$

We define the distribution

$$\langle T, \varphi\rangle = \int_1^\infty \Big[\varphi(\gamma_1(t)) - \varphi(\gamma_2(t))\Big]\, dt, \qquad \varphi \in C^\infty(\Omega),$$

then again this is an element of $\dot W^{-1,p}(\Omega)$, since by Sobolev embedding $[\varphi]_{C^{0,\alpha}} \leq C_\Omega \, \|\varphi\|_{\dot W^{1,q}}$, so that

$$\begin{aligned}
|\langle T, \varphi\rangle| \leq \int_1^\infty |\varphi(\gamma_1(t)) - \varphi(\gamma_2(t))|\, dt &\leq C\, \|\varphi\|_{W^{1,q}(\Omega)} \int_1^\infty |\gamma_1(t) - \gamma_2(t)|^\alpha\, dt \\
&= C\, \|\varphi\|_{W^{1,q}(\Omega)} \int_1^\infty \frac{\varepsilon^\alpha}{t^2}\, dt \\
&= C\, \varepsilon^\alpha\, \|\varphi\|_{W^{1,q}(\Omega)}, \qquad \varphi \in \dot W^{1,q}(\Omega).
\end{aligned}$$

We then introduce the measure on paths Q_T, defined by

$$Q_T = \int_1^\infty \delta_{\overline{\gamma_1(t)\,\gamma_2(t)}}\, dt,$$

where for every $t \geq 1$, by $\overline{\gamma_1(t)\,\gamma_2(t)}$ we indicate the straight segment going from $\gamma_1(t)$ to $\gamma_2(t)$. Observe that for every φ we have

$$\int_{\mathcal{L}(\overline{\Omega})} [\varphi(\gamma(0)) - \varphi(\gamma(1))]\, dQ_T(\gamma) = \int_1^\infty \Big[\varphi(\gamma_1(t)) - \varphi(\gamma_2(t))\Big]\, dt = \langle T, \varphi\rangle,$$

and

$$\begin{aligned}
\int_{\mathcal{L}(\overline{\Omega})} \ell(\gamma)\, dQ_T(\gamma) = \int_1^\infty |\gamma_1(t) - \gamma_2(t)|\, dt &= \int_1^\infty \frac{\varepsilon}{t^{2/\alpha}}\, dt \\
&= \varepsilon\, \frac{\alpha}{\alpha - 2}\, t^{1-\frac{2}{\alpha}}\Big|_1^\infty = \varepsilon\, \frac{\alpha}{2-\alpha} < \infty,
\end{aligned}$$

while Q_T is not finite, but just σ-finite.

The previous examples clarifies that we cannot hope to give a distributional meaning to the positive and negative parts of the divergence of V, and the good definition of $\operatorname{div} V$ as a distribution relies in general on some sort of "almost–cancellation". Therefore the last no-cancellation requirement $(\operatorname{div} V)_+ + (\operatorname{div} V)_- = (e_1 + e_0)_\# Q$ of Smirnov's Theorem A.14 must be relaxed when we extend it to a larger class of V's.

We introduce a notion needed in the next theorem:

Definition A.18. Let A, B, C be 1-currents. We say that C is a *subcurrent* of A if

$$\mathbb{M}(A - C) + \mathbb{M}(C) = \mathbb{M}(A).$$

In this case we write $C \leq A$. If also $\mathbb{M}(B-C)+\mathbb{M}(C) = \mathbb{M}(B)$ we say that C is a *common subcurrent of* A and B. We will use the notation

$$SC(A,B) = \{C \text{ is a } 1- \text{ current } : C \leq A \text{ and } C \leq B\},$$

to denote the set of common subcurrents.

Let $C \in SC(A,B)$, then we say that C is a *largest common subcurrent* of A and B if

$$SC(A-C, B-C) = \{0\},$$

where by 0 we denote the trivial $1-$current such that $\langle 0, \varphi\rangle = 0$ for every $1-$form φ. In this case, we will use the notation $C = A \wedge B$.

Lemma A.19. *Let A, B be normal 1-currents. Then they admit a largest common subcurrent C.*

Proof. The proof is exactly as in [27, Proposition 3.8] so we just indicate the main modifications needed in our case. For every pair of currents (T, S), we set

$$\xi(S,T) := \sup\{\mathbb{M}(C) \,:\, C \in SC(S,T)\}.$$

Let us define $\{A_\nu, B_\nu, C_\nu\}_{\nu\in\mathbb{N}}$ recursively as follows

$$\begin{cases} A_1 = A - C_0, \quad B_1 = B - C_0, \\ C_0 \in SC(A_0, B_0) \quad \text{ such that } \quad \mathbb{M}(C_0) \geq \dfrac{\xi(A_0,B_0)}{2}, \\ \\ A_{\nu+1} = A_\nu - C_\nu, \quad B_{\nu+1} = B_\nu - C_\nu \\ \text{where } C_\nu \in SC(A_\nu, B_\nu) \quad \text{ such that } \quad \mathbb{M}(C_\nu) \geq \dfrac{\xi(A_\nu, B_\nu)}{2} \qquad \text{for } \nu \geq 1. \end{cases}$$

Then as in [27] we have that $\mathbb{M}(C) \leq \xi(A_\nu, B_\nu)/2$ for all $C \in SC(A_{\nu+1}, B_{\nu+1})$ and in particular by taking the supremum we can infer

$$\xi(A_{\nu+1}, B_{\nu+1}) \leq \frac{\xi(A_\nu, B_\nu)}{2} \leq \cdots \leq \frac{\xi(A_0,B_0)}{2^{\nu+1}} \to 0.$$

From this we obtain that $\{A_\nu\}_{\nu\in\mathbb{N}}$ and $\{B_\nu\}_{\nu\in\mathbb{N}}$ are converging in mass norm and by setting

$$A' := \lim_{\nu\to\infty} A_\nu \qquad \text{and} \qquad B' := \lim_{\nu\to\infty} B_\nu,$$

we obtain $A' \leq A_\nu$ and $B' \leq B_\nu$, for every $\nu \in \mathbb{N}$. This implies that any $C' \in SC(A', B')$ also belongs to $SC(A_\nu, B_\nu)$ and by definition we have $\mathbb{M}(C') \leq \xi(A_\nu, B_\nu) \to 0$, thus we get $C' = 0$, i.e. the trivial current is the only common subcurrent of A' and B'. We then observe that $A - A' = B - B'$, since $A_{\nu-1} - A_\nu = B_{\nu-1} - B_\nu$ for $\nu \geq 1$ by construction, so we have

$$\begin{aligned} A - A' = \lim_{\nu\to\infty}[A - A_\nu] &= \lim_{\nu\to\infty}[C_0 + A_1 - A_\nu] \\ &= \lim_{\nu\to\infty}[C_0 + (A_1 - A_2) + \cdots + (A_{\nu-1} - A_\nu)] \\ &= \lim_{\nu\to\infty}[C_0 + (B_1 - B_2) + \cdots + (B_{\nu-1} - B_\nu)] \\ &= \lim_{\nu\to\infty}[B - B_\nu] = B - B'. \end{aligned}$$

We can finally set $C = A - A' = B - B' \in SC(A, B)$ and obtain $SC(A - C, B - C) = SC(A', B') = \{0\}$, i.e. C is a largest common subcurrent of A and B. $\square$

Theorem A.20 (Smirnov Theorem for flat acyclic currents)**.** *Let I be an acyclic flat 1-current, having finite mass. Then I is decomposable in curves, i.e. there exists a σ-finite positive measure Q on curves such that Q is concentrated on $\widetilde{\mathcal{L}}(\Omega)$ and*

$$I = \int_{\mathcal{L}(\Omega)} [\gamma] \, dQ(\gamma). \tag{A.6}$$

Moreover, we have

$$\mu_I = \int_{\mathcal{L}(\Omega)} \mu_{[\gamma]} \, dQ(\gamma) \tag{A.7}$$

and

$$\partial I = \int_{\mathcal{L}(\Omega)} \partial[\gamma] \, dQ(\gamma). \tag{A.8}$$

Proof. We start observing that thanks to Lemma A.7, our current I is approximable by normal currents $\{J_i\}_{i\in\mathbb{N}}$ in the mass norm, i.e. $\lim \mathbb{M}(I - J_i) = 0$ and $\mathbb{M}(J_i) + \mathbb{M}(\partial J_i) < \infty$. We construct a first decomposition of I using this information.

Step 1. *(I can be decomposed in normal currents).* We may assume that $\{J_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in the mass norm, i.e. that $\mathbb{M}(J_{i-1} - J_i) \le 2^{-i}$ for all i. We then define

$$\begin{cases} \tilde{I}_0 = J_{i_0}, & \\ \tilde{I}_k = J_{i_0+k} - J_{i_0+k-1} & \text{for } k \ge 1. \end{cases}$$

It then follows that $I = \sum_k \tilde{I}_k$ and $\sum_k \mathbb{M}(\tilde{I}_k) \le 2\,\mathbb{M}(I)$ if $i_0 \in \mathbb{N}$ is chosen large enough. We then modify the $\tilde{I}_k$ into a better decomposition I_k as follows:

$$\begin{cases} I_0 = \tilde{I}_0, & \\ I_{k+1} = \tilde{I}_{k+1} - \tilde{I}_{k+1} \wedge I_k & \text{for } k \ge 0. \end{cases},$$

where we recall that $\widetilde{I}_{k+1} \wedge I_k$ denotes a largest common subcurrent of $\widetilde{I}_{k+1}$ and I_k, which is well-defined thanks to Lemma A.19. From these definitions it follows that we have the decomposition

$$I = \sum_{k\ge 0} I_k, \qquad \mathbb{M}(I) = \sum_{k\ge 0} \mathbb{M}(I_k).$$

Moreover the currents I_k are still normal since each of them is a finite sum of subcurrents of normal currents.

Step 2. *(Use of Smirnov's result for normal currents).* By applying Theorem A.14 to each of the I_k we obtain a sequence of measures Q_k on the set $\mathcal{L}(\Omega)$ concentrated on $\widetilde{\mathcal{L}}(\Omega)$, such that Q_k decomposes I_k in the sense of Theorem A.14. From the previous step we

obtain then a decomposition for I as follows:

$$I = \sum_{k\geq 0} I_k = \sum_{k\geq 0} \int_{\mathcal{L}(\Omega)} [\gamma]\, dQ_k(\gamma), \tag{A.9}$$

and

$$\mathbb{M}(I) = \sum_{k\geq 0} \mathbb{M}(I_k) = \sum_{k\geq 0} \int_{\mathcal{L}(\overline{\Omega})} \ell(\gamma)\, dQ_k(\gamma). \tag{A.10}$$

STEP 3. *(Construction of a σ-finite measure).* We may then define a measure on $\mathcal{L}(\Omega)$ by setting

$$Q(A) = \sum_{k\geq 0} Q_k(A), \qquad \text{for every Borel set } A \subset \mathcal{L}(\Omega),$$

where the series on the right has to be intended as $\sum_{k\geq 0} Q_k(A) = \lim_{\nu\to\infty} \sum_{k=0}^{\nu} Q_k(A)$ and this limit exists by monotonicity (it is in fact a supremum), since the measures $\{Q_k\}_{k\in\mathbb{N}}$ are positive. At first, we verify that Q is indeed a measure: we have

$$Q(\emptyset) = \sum_{k=0}^{\infty} Q_k(\emptyset) = 0, \qquad \text{as } Q_k(\emptyset) = 0, \quad \text{for every } k \in \mathbb{N},$$

and also for any collection $\{A_i\}_{i=1}^{\infty}$ of disjoint Borel sets in $\mathcal{L}(\Omega)$ we get

$$\begin{aligned} Q\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{k=0}^{\nu} Q_k\left(\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{k=0}^{\infty}\sum_{i=1}^{\infty} Q_k(A_i) \\ &= \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} Q_k(A_i) = \sum_{i=0}^{\infty} Q(A_i), \end{aligned}$$

where we have the right to exchange the two series, since each term $Q_k(A_i)$ is positive. This finally shows that Q is a measure on $\mathcal{L}(\Omega)$. We also observe that

$$Q\left(\mathcal{L}(\Omega) \setminus \widetilde{\mathcal{L}}(\Omega)\right) = \sum_{k=0}^{\infty} Q_k\left(\mathcal{L}(\Omega) \setminus \widetilde{\mathcal{L}}(\Omega)\right) = 0,$$

as each Q_k is concentrated on $\widetilde{\mathcal{L}}(\Omega)$.

In order to verifiy that Q is $\sigma-$finite, for all $h \in \mathbb{Z}$ we define the Borel sets $B^h := \{\gamma \in \mathcal{L}(\Omega) : \ell(\gamma) \in [2^h, 2^{h+1}[\}$, which form a partition of $\mathcal{L}(\Omega)$. Observe that the fact that B^h is a Borel set easily follows from the lower semicontinuity of the length functional. Moreover, for fixed $h \in \mathbb{Z}$ there holds

$$Q(B^h) = \sum_{k\geq 0} Q_k(B^h) \leq 2^{-h} \sum_{k\geq 0} \mathbb{M}(I_k) = 2^{-h}\, \mathbb{M}(I) < +\infty.$$

This finally shows that Q is $\sigma-$finite. The fact that Q verifies (A.6) and (A.7) follows directly from (A.9) and (A.10), while (A.8) is a consequence of (A.6), which suffices to test on special forms $\omega = d\phi$. $\square$

Remark A.21. The proof of the Theorem A.20 extends to general currents on metric spaces, following [27] (see also [28] and [36]).

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